

Group Members

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Problem Set 1

1. • $f(x, y) = x^a y^b$

a) $\frac{\partial f}{\partial x} = a x^{a-1} y^b$ b) $\frac{\partial f}{\partial y} = b x^a y^{b-1}$ c) $\frac{\partial f}{\partial z} = 0$

• $f(x, y) = (ax^\delta + by^\delta)^{\frac{1}{\delta}}$

a) $\frac{\partial f}{\partial x} = \frac{1}{\delta} (ax^\delta + by^\delta)^{\frac{1}{\delta}-1} \cdot \delta ax^{\delta-1} = (ax^\delta + by^\delta)^{\frac{1}{\delta}-1} \cdot ax^{\delta-1}$

b) $\frac{\partial f}{\partial y} = \frac{1}{\delta} (ax^\delta + by^\delta)^{\frac{1}{\delta}-1} \cdot \delta by^{\delta-1} = (ax^\delta + by^\delta)^{\frac{1}{\delta}-1} \cdot by^{\delta-1}$

c) $\frac{\partial f}{\partial z} = 0$

• $f(x, y, z) = \frac{a^a b^b}{(a+b)^{a+b}} \frac{z^{a+b}}{x^a y^b}$

a) $\frac{\partial f}{\partial x} = \frac{a^a b^b}{(a+b)^{a+b}} \frac{z^{a+b}}{y^b} \cdot (-a) x^{-a-1}$

b) $\frac{\partial f}{\partial y} = \frac{a^a b^b}{(a+b)^{a+b}} \frac{z^{a+b}}{x^a} \cdot (-b) y^{-b-1}$

c) $\frac{\partial f}{\partial z} = \frac{a^a b^b}{(a+b)^{a+b}} \frac{1}{x^a y^b} \cdot (a+b) z^{a+b-1}$

$$\bullet f(x, y, z) = \frac{z}{(ax^{\eta-1} + by^{\eta-1})^{\frac{1}{1-\eta}}} \quad \text{also written as} \quad z(ax^{\eta-1} + by^{\eta-1})^{\frac{1}{\eta-1}}$$

$$a) \frac{\partial f}{\partial x} = \frac{z}{\eta-1} (ax^{\eta-1} + by^{\eta-1})^{\frac{1}{\eta-1}-1} \cdot (\eta-1) ax^{\eta-2} = zax^{\eta-2} (ax^{\eta-1} + by^{\eta-1})^{\frac{1}{\eta-1}-1}$$

$$b) \frac{\partial f}{\partial y} = \frac{z}{\eta-1} (ax^{\eta-1} + by^{\eta-1})^{\frac{1}{\eta-1}-1} \cdot (\eta-1) by^{\eta-2} = zby^{\eta-2} (ax^{\eta-1} + by^{\eta-1})^{\frac{1}{\eta-1}-1}$$

$$c) \frac{\partial f}{\partial z} = (ax^{\eta-1} + by^{\eta-1})^{\frac{1}{\eta-1}}$$

$$\bullet f(x, y) = \sqrt{[\ln(x) + \ln(y)]^2 + 7} = \left[[\ln(x) + \ln(y)]^2 + 7 \right]^{\frac{1}{2}}$$

$$a) \frac{\partial f}{\partial x} = \frac{1}{2} \left[[\ln(x) + \ln(y)]^2 + 7 \right]^{-\frac{1}{2}} \cdot 2[\ln(x) + \ln(y)] \cdot \frac{1}{x} = \frac{\ln(x) + \ln(y)}{x \sqrt{[\ln(x) + \ln(y)]^2 + 7}}$$

$$b) \frac{\partial f}{\partial y} = \frac{1}{2} \left[[\ln(x) + \ln(y)]^2 + 7 \right]^{-\frac{1}{2}} \cdot 2[\ln(x) + \ln(y)] \cdot \frac{1}{y} = \frac{\ln(x) + \ln(y)}{y \sqrt{[\ln(x) + \ln(y)]^2 + 7}}$$

$$c) \frac{\partial f}{\partial z} = \frac{1}{2} \left[[\ln(x) + \ln(y)]^2 + 7 \right]^{-\frac{1}{2}} \cdot 2[\ln(x) + \ln(y)] \cdot 0 = 0$$

$$2. \quad f(x, y) = 4x^2 + 3y^2$$

$$a) \frac{\partial f}{\partial x} = 8x + 0 = 8x$$

$$\frac{\partial f}{\partial y} = 0 + 6y = 6y$$

$$b) \frac{\partial f}{\partial x} \Big|_{x=1} = 8(1) = 8$$

$$\frac{\partial f}{\partial y} \Big|_{y=2} = 6(2) = 12$$

$$c) df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$df = 8x dx + 6y dy$$

$$d) \frac{dy}{dx} \text{ such that } df = 0$$

$$0 = 8x dx + 6y dy$$

$$6y dy = -8x dx$$

$$dy = \frac{-8x dx}{6y} \quad \frac{dy}{dx} = \frac{-4x}{3y}$$

2. continued) e) $f=16$ such that $x=1, y=2$

$$f(1,2) = 4(1)^2 + 3(2)^2 = 4 + 12 = 16$$

$$f = 16$$

f) $\frac{dy}{dx} = \frac{-4x}{3y}$ (from part d)

$$x=1 \quad y=2 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-4(1)}{3(2)} = \frac{-4}{6} = -\frac{2}{3}$$

g) $f(x,y) = 4x^2 + 3y^2$

slope of contour line @ $f=16$

$$y^2 = \frac{16 - 4x^2}{3}$$

$$y = \left(\frac{16 - 4x^2}{3} \right)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{16 - 4x^2}{3} \right)^{-1/2} \cdot \left(-\frac{4}{3}(2)x \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{16 - 4x^2}{3} \right)^{-1/2} \cdot \left(-\frac{8}{3}x \right) = -\frac{4x}{3} \left(\frac{16 - 4x^2}{3} \right)^{-1/2}$$

3 a) $\max_{x,y} xy$
s.t. $x+y=1$

$$1-x-y=0$$

$$c-x-y=0$$

(constraint)

$$L = xy + \lambda(1-x-y)$$

$$\frac{\partial L}{\partial x} = y - \lambda = 0$$

$$y^* = \lambda$$

$$\frac{\partial L}{\partial y} = x - \lambda = 0$$

$$x^* = \lambda$$

$$x^* = y^*$$

$$\frac{\partial L}{\partial \lambda} = 1-x-y=0$$

$$1 = 2x^*$$

$$\frac{1}{2} = x^*$$

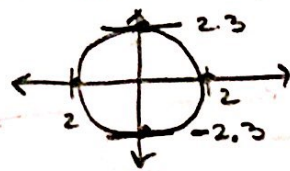
$$\frac{1}{2} = y^*$$

$$\lambda^* = \frac{1}{2}$$

side

$$\lambda \frac{\Delta f}{\Delta c}$$

The general shape is an ellipse.



$$16 - 4x^2 + 3y^2$$

The Lagrangean Multiplier indicates the relationship between the objective and the constraint. Such as how much more value can I receive from relaxing the constraint. Here it tells me, $\lambda^* = \frac{1}{2}$ that I can get 1 more unit of area (xy) by relaxing the constraint by 2 units.

3. continued

$$b) \min_{x,y} x + y$$

$$s.t. \quad xy = 0.25$$

$$\rightarrow 0.25 - xy = 0$$

$$L = x + y + \lambda(0.25 - xy)$$

$$\frac{\partial L}{\partial x} = 1 - \lambda y = 0$$

$$\lambda y^* = 1$$

$$y^* = \frac{1}{\lambda^*}$$

$$\frac{\partial L}{\partial y} = 1 - \lambda x = 0$$

$$\lambda x^* = 1$$

$$x^* = \frac{1}{\lambda^*}$$

$$\frac{\partial L}{\partial \lambda} = 0.25 - xy = 0$$

$$0.25 = x^* y^*$$

$$\rightarrow x^* = y^* \rightarrow \sqrt{0.25} = \sqrt{x^2} \text{ or } \sqrt{0.25} = \sqrt{y^2}$$

$$-0.5 = x^* \quad -0.5 = y^*$$

$$\lambda^* = -2$$

* choose negative because we want to minimize $x + y$.

The Lagrange multiplier, λ , here shows that by changing the constraint of 0.25 by 1, the change in the min value of the objective function would change by 2.

$$\begin{aligned} & -0.5 + -0.5 \\ & = -0.5 - 0.5 \\ & = -1 \end{aligned}$$

4. $f(t) = -0.5gt^2 + 40t$

$$a) \frac{df}{dt} = -gt + 40 = 0$$

$$gt^* = 40$$

$$t^* = \frac{40}{g}$$

The value of t^* depend on parameter g because the greater the value of g , the smaller the value of t^* as a result.

$$b) f(t^*) = -0.5g\left(\frac{40}{g}\right)^2 + 40\left(\frac{40}{g}\right)$$

$$= -\frac{800g}{g^2} + \frac{1600}{g} = \frac{800}{g}$$

$$\frac{df(t^*)}{dg} = -\frac{800}{g^2}$$

4. continued

$$c) \frac{df(t^*)}{dg} = \frac{\partial f(t^*)}{\partial g} = -0.5t^2 \Big|_{t=t^*} = -0.5\left(\frac{40}{g}\right)^2$$

$$= \frac{-800}{g^2}$$

$$d) \frac{\partial f(t)}{\partial g} = \frac{-800}{g^2} \quad \partial f(t) = -\frac{800}{32^2} \partial g$$

$$g=32 \quad = -\frac{800}{1024} (0.1) \approx -0.078125$$

$$\partial f(t) \approx 0.08$$

* lifted the negative since the problem is asking for the change.

5.

$$\max_{x_1, x_2} \quad y = x_1 + 5 \ln(x_2) \quad \text{s.t.} \quad k - x_1 - x_2 = 0$$

$$a) \text{ Let } k=10 \quad L(x_1, x_2, \lambda) = x_1 + 5 \ln(x_2) + \lambda(10 - x_1 - x_2)$$

$$\frac{\partial L}{\partial x_1} = 1 - \lambda = 0 \quad \frac{\partial L}{\partial x_2} = \frac{5}{x_2} - \lambda = 0 \quad \frac{\partial L}{\partial \lambda} = 10 - x_1 - x_2 = 0$$

$$1 = \lambda^* \quad 10 = x_1^* + x_2^*$$

$$\frac{5}{x_2} = \lambda^*$$

$$10 = x_1^* + 5$$

$$x_1^* = 5$$

$$\frac{5}{\lambda^*} = x_2^*$$

$$x_1^* = 5 \quad x_2^* = 5$$

$$5 = x_2^*$$

$$b) \quad k=4$$

$$L(x_1, x_2, \lambda) = x_1 + 5 \ln(x_2) + \lambda(4 - x_1 - x_2)$$

$$\lambda^* = 1$$

$$x_2^* = \frac{5}{\lambda^*} = 5$$

$$\frac{\partial L}{\partial x_1} = 1 - \lambda = 0$$

$$\frac{\partial L}{\partial x_2} = \frac{5}{x_2} - \lambda = 0$$

$$\frac{5}{x_2^*} = \lambda^*$$

$$\frac{\partial L}{\partial \lambda} = 4 - x_1 - x_2 = 0$$

$$x_1^* = -1$$

5. continued

c) x 's must be non negative, $\therefore$ add constraint : $x_1 \geq 0$

$$y = x_1 + 5 \ln(x_2) \quad \text{st.} \quad 4 - x_1 - x_2 \geq 0$$

$$x_1 \geq 0$$

$$x_2 \geq 0 \quad \text{Don't need constraint for } x_2 \text{ due to the behavior of } \ln(x_2)$$

$$L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3) = x_1 + 5 \ln(x_2) + \lambda_1(4 - x_1 - x_2) + \lambda_2(x_1)$$

$$\frac{\partial L}{\partial x_1} = 1 - \lambda_1 + \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_2} = \frac{5}{x_2} - \lambda_1 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = 4 - x_1 - x_2 = 0$$

$$4 - 0 - x_2^* = 0$$

$$4 = x_2^*$$

$$\frac{\partial L}{\partial \lambda_2} = x_1^* = 0$$

$$0 = x_1^*$$

d) $k=20 \quad \max_{x_1, x_2} y = x_1 + 5 \ln(x_2) \quad \text{st} \quad 20 - x_1 - x_2 = 0$

$$L(x_1, x_2, \lambda_1) = x_1 + 5 \ln(x_2) + \lambda_1(20 - x_1 - x_2)$$

$$\frac{\partial L}{\partial x_1} = 1 - \lambda_1 = 0$$

$$1 = \lambda_1^*$$

$$\frac{\partial L}{\partial x_2} = \frac{5}{x_2} - \lambda_1 = 0$$

$$\frac{5}{x_2^*} = \lambda_1^*$$

$$\frac{5}{1} = x_2^*$$

$$x_2^* = 5$$

$$\frac{\partial L}{\partial \lambda_1} = 20 - x_1 - x_2 = 0$$

$$20 - x_2^* = x_1^*$$

$$20 - 5 = x_1^*$$

$$x_1^* = 15$$

concluded That while the optimal value x_2^* stays the same, x_1^* increases as constraint $\uparrow$'s.

5. continued

$$c) \quad v(k) = \max_{x_1, x_2} x_1 + 5 \ln(x_2) \quad \text{s.t.} \quad k - x_1 - x_2 \geq 0$$

$$L(x_1, x_2, \lambda_1) = x_1 + 5 \ln(x_2) + \lambda_1 (k - x_1 - x_2)$$

$$\frac{\partial L}{\partial x_1} = 1 - \lambda_1^* = 0$$

$$\lambda_1^* = 1$$

$$\frac{\partial L}{\partial x_2} = \frac{5}{x_2} - \lambda_1 = 0$$

$$\frac{5}{x_2^*} = \lambda_1^*$$

$$x_2^* = \frac{5}{1} = 5$$

$$\frac{\partial L}{\partial \lambda_1} = k - x_1 - x_2 = 0$$

$$\frac{dv}{dk} = \frac{\partial L}{\partial k} \Big|_{x_1=x_1^*, x_2=x_2^*, \lambda_1=\lambda_1^*}$$

$$\frac{dv}{dk} = \lambda_1 \Big|_{\lambda_1=\lambda_1^*}$$

$$\frac{dv}{dk} = \lambda_1^*$$

$$\frac{dv}{dk} = 1$$

Interpreted That as you relax the constraint by a marginal unit of 1, the optimal value of $v(k)$, increases by 1.

6.

Students: $\max_h U_i(h, d) = -\frac{d}{h} + f_i(h)$

Teacher: $\max_d U_t = 29d + \sum_{i=1}^{348} U_i(h^*, d)$

Solve for d^*

$$\frac{d U_t}{d d} = 29 + \sum_{i=1}^{348} \frac{d U_i(h^*, d)}{d d} \Big|_{h=h^*}$$

(envelope Theorem)

$$= 29 + \sum_{i=1}^{348} \frac{\partial U_i}{\partial d} \Big|_{h=h^*}$$

side: $\frac{\partial U_i}{\partial d} = -\frac{1}{h}$

$$= 29 + \sum_{i=1}^{348} \frac{-1}{h^*}$$

$$= 29 + \frac{-348}{4d^{\frac{1}{2}}}$$

$$h^* = 4d^{\frac{1}{2}}$$

(set equal to 0 to find d^*)

$$0 = 29 + \frac{-348}{4d^{\frac{1}{2}}}$$

$$29 = \frac{348}{4d^{\frac{1}{2}}}$$

$$d^{\frac{1}{2}} = \frac{348}{29 \cdot 4}$$

$$d^* = \left(\frac{348}{116}\right)^2$$

$$d^* = \frac{121,104}{13,456}$$

$$d^* = 9$$